Minimum Weight Design of Complex Structures Subject to a Frequency Constraint

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An analytical procedure is presented for the determination of the least weight structure which satisfies a specific frequency requirement. The approach is to modify an existing structure by varying the cross-sectional properties of its member. This is accomplished using separate gradient equations to first obtain the correct structural frequency and then, while the frequency is held constant, to minimize the weight. These equations are derived in matrix notation for direct application to the finite element method of analysis. This procedure has been completely automated in a computer program which includes the substructure analysis capability for treating large complex structures. Problems up to 1800 degrees of freedom may be optimized by varying as many as 100 of the structural elements in a single run. Results of four numerical examples show that the method is convergent and that optimized configurations can be determined in less than 10 redesign cycles.

1. Introduction

In many spacecraft, all or part of the structure may be designed to meet a minimum frequency requirement, i.e., the fundamental frequency must be above some specified level. Along with this requirement, which is often imposed to avoid coupling with the booster control system, there exists the sometimes overriding need to minimize the spacecraft weight. For the sake of brevity, the process of satisfying both of these constraints is designated "dynamic optimization."

Numerous publications have appeared in the literature[1–4] dealing with the dynamic optimization of simple structures. The most common example treated is the longitudinal vibration of a bar, fixed at one end and having a tip mass at the other end. The analytical techniques presented are, however, not amenable to the solution of complex structural problems. Turner[1] attempted to generalize his approach using matrix formulation. The governing equations are expressed as a system of nonlinear equations which are solved by an iterative procedure which is an adaptation of the Newton-Raphson method. Application of the method requires that the analyst provide an initial estimate of the fundamental mode. Some difficulty in convergence may be encountered if the mode shape should change drastically during the iterative process. In an earlier publication, Young and Christiansen[5] optimized a space truss. They employed a mixed static-dynamic analysis in that the structure was termed optimum if it was uniformly stressed when the static deflection was given by the mode shape arbitrarily normalized to unit vector length. The authors point out, however, that instabilities may be introduced in the redesign process depending upon the manner in which the mode is normalized. Another drawback to this approach is that a fully stressed design does not necessarily lead to a minimum weight design.

In a more recent publication Zarghamee[6] presented a method for the maximization of the fundamental frequency of a complex structure having a fixed weight. The procedure is based on gradient equations which express the rate of change of the frequency with respect to the design parameters. These equations were also developed by Fox and Kapoor.[7] The gradients can be easily evaluated in terms of the element stiffness and mass matrices and the associated mode shape. As a result, the determination of the incremental structural modifications is straightforward and enables an efficient redesign process to be established.

The approach presented herein utilizes similar gradient equations for application to dynamic optimization. A stepwise procedure is employed and is defined in terms of two modes of travel: a frequency-modification mode and a weight-minimization mode, both based on the previous gradient equations. The initial steps are taken in the frequency-modification mode until the desired frequency is reached. The weight-minimization mode is then used to reduce the weight while the frequency of the structure is maintained.

A large-scale computer program was developed to completely automate the aforementioned approach. The parameters which are varied during the redesign process are the thickness of structural elements configured as thin walled circular tubes with shear deformation. These elements are capable of representing simple tubes as well as thin-walled shells with bending effects neglected. In order to make the computer program suitable for the analysis of large complex structures, the capability of performing substructure analysis by the method of component mode synthesis[8,9] was included. Numerical results are presented to demonstrate both the character and the accuracy of the redesign process.

2. Optimization Procedure

The process for dynamic optimization is defined in terms of two modes of travel: 1) the frequency-modification mode which is used to adjust the fundamental frequency of the structure to some specified value, and 2) the weight-minimization mode which is designed to reduce the structural weight while maintaining the desired frequency. It is assumed that the optimum design being sought lies on the frequency constraint, i.e., higher frequency designs will necessarily be heavier. To compensate for any frequency drift which may occur during the weight-minimization mode, an occasional step in the frequency-modification mode is introduced to return the fundamental frequency to the desired value.

The equations governing the redesign process in both modes of travel are based on the gradient equations which were first published by Zarghamee[6] but were derived inde-

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pendently for the present analysis. The formulation of these equations in the following section provides the basic material for the development of the redesign process.

Gradient Equations

The derivation which follows is general in the sense that it applies to any natural frequency. For the purposes of this paper, however, it will later be restricted to the fundamental frequency of the structure.

The equations of motion of a structure modeled with discrete elements and assembled by the finite element technique, are expressed in matrix form as follows:

$$[K]\{u\} - \lambda[M]\{u\} = 0 \tag{1}$$

in which $[K]$ and $[M]$ are the system stiffness and mass matrices, respectively, $[u]$ is the mode vector, and λ is the square of the corresponding natural frequency. The system matrices are formed in the usual manner by summing the stiffness and mass matrices of each element.

It is assumed in the present analysis that some or possibly all of the elements will be allowed to vary in their geometry in order to meet the frequency and minimum weight conditions. Only one parameter is allowed to vary per element and this parameter must occur as a linear factor in the element matrices. Under these conditions the system stiffness and mass matrices may be written in the following form:

$$\begin{aligned} [K] &= \sum_{j=1}^{n} a_j[K_j] + [\bar{K}] \\ [M] &= \sum_{j=1}^{n} a_j[M_j] + [\bar{M}] \end{aligned} \tag{2}$$

where n is the number of variable elements and a_j are the variable parameters. Note that the products $a_j[K_j]$ and $a_j[M_j]$ are the jth element stiffness and mass matrices and $[\bar{K}]$ and $[\bar{M}]$ represent the stiffness and mass characteristics of that portion of the structure which is considered nonvariable.

If Eq. (1) is now differentiated with respect to a_j, by Eq. (2) the result may be written as follows:

$$[K]\{u,_{a_j}\} - \lambda[M]\{u,_{a_j}\} = \lambda,_{a_j}[M]\{u\} - [K_j]\{u\} + \lambda[M_j]\{u\} \tag{3}$$

where the comma represents partial differentiation with respect to the indicated variable. Multiplication of this equation by $\{u\}^T$makes the left hand side equal to zero due to the symmetry of $[K]$ and $[M]$ and the satisfaction of Eq. (1). The equation therefore becomes

$$0 = \lambda,_{a_j}\{u\}^T[M]\{u\} - \{u\}^T([K_j] - \lambda[M_j])\{u\} \tag{4}$$

For convenience, the generalized mass is assumed equal to one, thus enabling this equation, designated the basic gradient equation, to be written in the form:

$$\lambda,_{a_j} = G_j \tag{5}$$

where

$$G_j = \{u\}^T([K_j] - \lambda[M_j])\{u\} \tag{6}$$

The importance of Eq. (5), as pointed out in Ref. 6, is that it provides a simple and direct method for calculating the rate of change of the square of the frequency with respect to each of the element parameters in terms of the current value of the frequency and its corresponding mode shape.

Frequency-Modification Mode

The optimization procedure, described herein, first operates on the structure to change its fundamental frequency to some specified value. This phase of the redesign process occurs in the "frequency-modification" mode. In the discussion of the governing equations which follow, the procedure is restricted to modification of the fundamental frequency and to variable parameters which are given by the thicknesses of the structural elements. Structural elements which have their thickness as a linear factor in both the stiffness and mass matrices include thin-walled tubular beams and plates of sandwich construction. Although the equations have more general application, it is felt that restriction of the discussion lends more clarity to the features of this approach.

There are several possible approaches one may take in defining the redesign process. In the first method considered, the changes in thickness are taken proportional to the gradient, G_i, defined in Eq. (6):

$$\Delta t_i = KG_i \tag{7}$$

where K is a constant. The corresponding increase in the change in the square of the frequency is given by

$$\Delta\omega^2 = \sum_{i=1}^{n} G_i\Delta t_i = K \sum_{i=1}^{n} G_i^2 \tag{8}$$

For a prescribed change in frequency, the constant K may be evaluated as follows:

$$K = \Delta\omega^2 \Big/ \sum_{i=1}^{n} G_i^2 \tag{9}$$

The resulting change in structural weight is determined from the equation

$$\Delta W = \sum_{i=1}^{n} w_i\Delta t_i \tag{10}$$

where w_i is the weight coefficient and w_it_i represents the weight of the ith element. Substitution of Eqs. (7) and (9) into (10) leads to

$$\Delta W = \Delta\omega^2 \sum_{i=1}^{n} w_iG_i \Big/ \sum_{i=1}^{n} G_i^2 \tag{11}$$

An alternate approach was considered which appeared to have the potential of reducing the increase in weight which would result from an increase in the frequency without substantially reducing the rate of frequency change. The difference lies in the selection of the gradient in the form G_i/w_i rather than simply G_i. The basis for this selection came from an examination of the following variational problem:

$$\text{minimize } W = \sum_{i=1}^{n} W_i \tag{12}$$

$$\text{subject to } \lambda = \lambda_{\text{Desired}}$$

In terms of a Lagrangian multiplier β, this becomes:

$$\text{minimize } W - \beta(\lambda - \lambda_{\text{Desired}}) \tag{13}$$

The variations with respect to the thicknesses, t_i, yield the following Euler equations:

$$\partial W_i/\partial t_i - \beta(\partial\lambda_i/\partial t_i) = 0$$

which, by Eq. (5) and the relationship $W_i = w_it_i$, leads to

$$G_i/w_i = 1/\beta \tag{14}$$

Thus, the optimum configuration which satisfies both the frequency and minimum weight conditions has G_i/w_i equal to the same constant for every variable element. Thickness changes taken proportional to G_i/w_i would, therefore, cause the redesign process to approach the optimum conditions by continually reducing the larger values of G_i/w_i such that the spread between extreme values is also reduced.

The total weight change for this method, with the change in frequency prescribed, can be shown as above to take the following form:

$$\Delta W^* = \left(\Delta\omega^2 \sum_{i=1}^{n} G_i\right) \Big/ \left(\sum_{i=1}^{n} G_i^2/w_i\right) \qquad (15)$$

A comparison of the weight changes of the two methods, given by the ratio

$$\frac{\Delta W}{\Delta W^*} = \frac{\sum_{i=1}^{n} w_i G_i \sum_{i=1}^{n} \frac{G_i^2}{w_i}}{\sum_{i=1}^{n} G_i^2 \sum_{i=1}^{n} G_i} \qquad (16)$$

leads to the following results.

1) If all the w_i are equal then the preceding ratio is equal to one and both methods are identical.

2) If all the G_i/w_i are equal, i.e., the current configuration is optimum for that frequency, then again the ratio becomes one.

For the general case, however, it has not been possible to prove, as speculated, that the ratio is always greater than or equal to one.

Since both approaches seem equally acceptable it was decided arbitrarily to employ the first which then makes the frequency-modification mode identical to the method of steepest descent. The specific redesign equations selected for this analysis are summarized below:

$$\Delta t_i = KG_i \qquad (17)$$

if the frequency is to be increased and

$$\Delta t_i = Kt_i \qquad (18)$$

if the frequency is to be decreased. The uniform thickness change given by Eq. (18) was also chosen arbitrarily since the most suitable path in the decreasing frequency problem would lead to a slow rate of change in the frequency. This condition occurs because it is desired to obtain the greatest reduction in weight for the least change in frequency. This approach seems to be a reasonable compromise and is simple to implement.

The stepsize for each redesign cycle, given by the constant K in Eqs. (17) and (18), is determined such that

$$\Delta\omega^2/\omega^2 \leq \gamma \qquad (19)$$

and

$$\Delta t_i/t_i \leq \gamma \text{ for all } i \qquad (20)$$

where γ is some prescribed constant. To remain within the validity of the redesign equations, which are only incremental expressions, it is necessary to have $\gamma \ll 1$. However, a more efficient scheme is to initially select γ larger, say close to one, and check the resulting frequency after the first cycle. If the change in frequency is less than that predicted by the linearized Eq. (8), then γ should be reduced for the next step. If, on the other hand, fairly good agreement is obtained, then the large stepsize has succeeded in providing a rapid rate of convergence to the desired frequency.

Weight-Minimization Mode

The "weight-minimization" mode is employed after the desired frequency has been reached. Its function is to minimize the weight of the variable elements while maintaining the frequency level. The frequency constraint may be expressed as follows:

$$\Delta\omega^2 = \sum_{i=1}^{n} G_i \Delta t_i = 0 \qquad (21)$$

This equation states that $n - 1$ of the elements may be varied independently, whereas, the final element must satisfy:

$$\Delta t_m = -\left(\sum_{\substack{i=1\\ i\neq m}}^{n} G_i \Delta t_i\right) \Big/ G_m \qquad (22)$$

The objective of this phase of the redesign process is to maximize the negative change in weight of the variable elements, i.e., to maximize

$$-\Delta W = -\sum_{i=1}^{n} w_i \Delta t_i \qquad (23)$$

Substitution of Eq. (22) into Eq. (23) yields

$$-\Delta W = -\sum_{\substack{i=1\\ i\neq m}}^{n} w_i \Delta t_i + \frac{w_m}{G_m} \sum_{\substack{i=1\\ i\neq m}}^{n} G_i \Delta t_i = \sum_{\substack{i=1\\ i\neq m}}^{n} g_i \Delta t_i \qquad (24)$$

where

$$g_i = (w_m G_i - w_i G_m)/G_m \qquad (25)$$

The g_i are the components of the gradient of $(-\Delta W)$. At the optimum configuration the g_i will be equal to zero and

$$G_i/w_i = G_m/w_m \qquad (26)$$

or, in other words, all the G_i/w_i will equal the same constant. This agrees with the result obtained in Eq. (14).

Again, by the method of gradients, the thickness changes are defined as follows:

$$\Delta t_i = Cg_i \text{ for } \underset{i\neq m}{i = 1} \text{ to } n \qquad (27)$$

where C is a constant. The mth thickness change is given by Eq. (22).

The total change in weight is, therefore, given by

$$-\Delta W = \sum_{\substack{i=1\\ i\neq m}}^{n} g_i \Delta t_i = C \sum_{\substack{i=1\\ i\neq m}}^{n} g_i^2 \qquad (28)$$

The mth element is selected as the element which, when removed from the aforementioned summation, gives the greatest weight change in Eq. (28). Note that the step-size constant, C, must remain fixed during the comparative evaluation.

There are many ways to select the value of the constant C. In the computer application of this procedure C is initialized to a value which satisfies conditions similar to those presented in Eqs. (19) and (20). The value is reduced in the next step if the new fundamental frequency represents an unacceptably large drift away from the desired frequency.

3. Computer Program Description

The optimization procedure described in the previous sections was implemented in a large scale computer program for the analysis of complex structures. A flow diagram of the redesign process is presented in Fig. 1. The program is capable of performing the complete dynamic optimization or simply to terminate redesign after the desired frequency is reached. Intermediate checks of the results can be obtained by specifying only a small number of allowable cycles. The analysis may then be restarted by means of a single data card.

Each redesign cycle consists of the following steps: 1) determine thickness changes based on gradient expressions evaluated from the current configuration, 2) form new elemental stiffness and mass matrices and reassemble the total system stiffness and mass matrices, and 3) solve the eigenvalue problem to determine the fundamental frequency and mode of the new configuration. The last step is indeed the most time consuming but was felt to be necessary in order to take into account the possibility of a drastic change in the fundamental mode. This might occur,

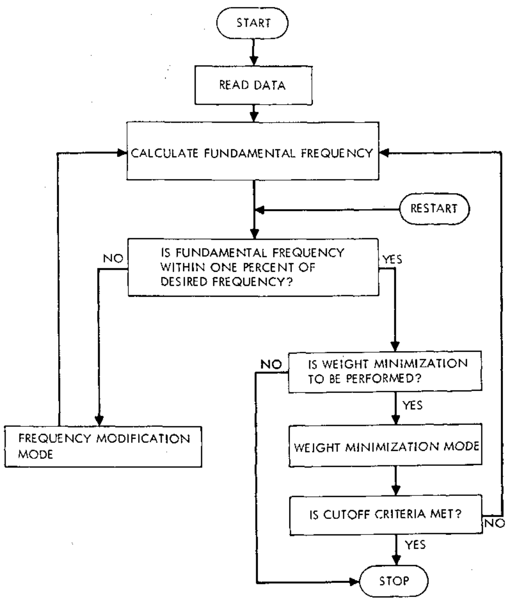

Fig. 1 Flow diagram of the dynamic optimization computer program.

for example, if the structure has a number of frequencies near the fundamental. The more efficient approximate techniques which simply add incremental changes to the current mode would, for example, be unable to detect a shift in the order of the modes which might occur as a result of the redesign step.

The redesign process is automatically terminated if the following two conditions are met. 1) the frequency is within one percent of the desired value, and 2) the change in weight determined for the next cycle is less than or equal to some prescribed fraction of the current total weight of the variable structural elements.

An important feature of this computer program is its capability to perform a substructure analysis by the method of component mode synthesis. In this approach a structure is subdivided into components, each of which is analyzed separately. Two types of component analyses may be performed: in one the interface points are left free while in the other these points are fixed. The latter method was selected for the computer analysis. The total system dynamic behavior is

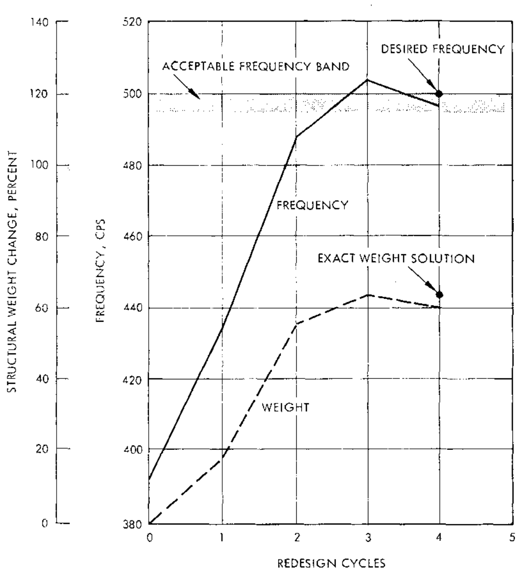

Fig. 2 Optimization of a uniform bar, increased frequency example.

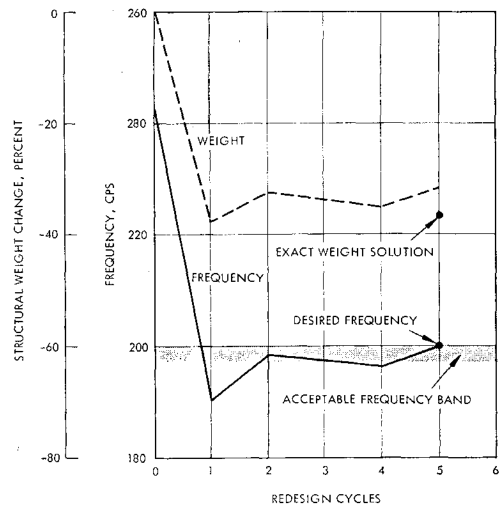

Fig. 3 Optimization of a uniform bar, decreased frequency example.

determined by coupling together a limited number of "fixed" modes from each component. The purpose of the modal truncation is to make the order of the final synthesized stiffness and mass matrices as small as possible. If only the lower frequencies and modes of the complete structure are desired, then a considerable reduction in the number of degrees of freedom can be realized.

The equations employed in the analysis are extremely similar to those formulated by Craig and Bampton[9] and, thus, will not be repeated here. The principal difference between these analyses and the more widely known approach of Hurty[8] is that the rigid body and constraint modes are treated alike. Hurty defines the constraint modes as the component displacements which are produced by unit displacements at each of the redundant interface points. The advantage of the present method is that it does not require the engineer to make the distinction between statically determinant and redundant freedoms.

The computer program has the capability of analyzing a structure comprised of as many as ten components. Each component is analyzed using lumped masses and may have as many as 180 degrees of freedom. A maximum size problem of approximately 1800 degrees of freedom is, therefore, possible. The dynamic analysis of each component is performed using a version of the computer program described in Ref. 10. This program is linked together with the component mode synthesis capability and the dynamic optimization procedure to form a single program to completely automate the redesign process.

For multiple component problems, computational efficiency made it desirable to restrict variable elements to one component only. It is not necessary then to recompute the modal characteristics of the remaining components during the iteration process. Up to 100 variable elements may be treated at one time. The variable element employed in this computer program is a thin walled tubular element that includes shear deformation. This element may be either cylindrical or conical and may have honeycomb sandwich construction, in which case the face sheet thickness is the variable parameter.

4. Numerical Results

Several numerical examples are presented to demonstrate the feasibility of the dynamic optimization procedure. The first two examples were used to evaluate the accuracy of the approach. Two different uniform aluminum bars, each 88 in. in length were fixed at one end and had a 10-pound tip mass at the other end. Both bars were divided into eleven equal length tubular elements. In the first example, each element

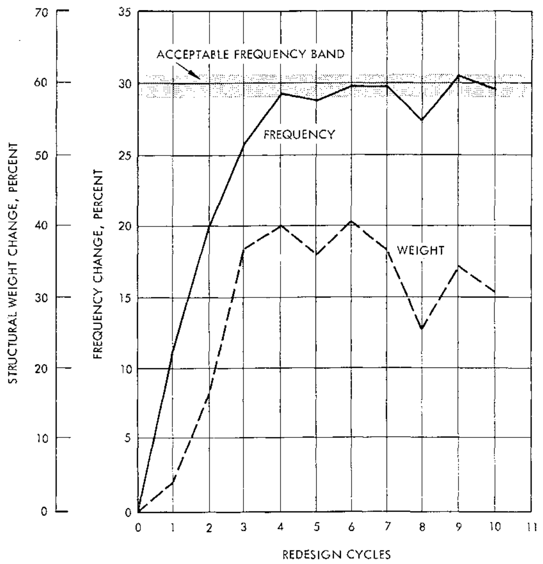

Fig. 4 Optimization of a frame structure having 126 degrees of freedom.

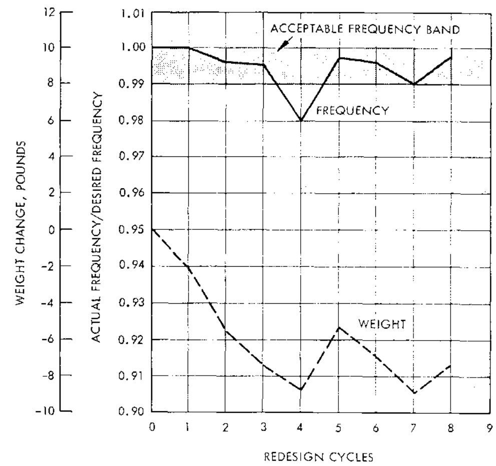

Fig. 5 Weight minimization of a structure having 246 degrees of freedom.

had a radius of 5 in. and an initial thickness of 0.08 in. The objective was to minimize the weight while increasing the fundamental frequency from 393 to 500 cps. The results of dynamic optimization after each redesign cycle are shown in Fig. 2. Also shown in the graph is the exact solution which was obtained from the equations derived by Turner.[1] The shaded area represents the acceptable frequency band. This example shows convergence to a structural weight which is in close agreement with the exact solution. Due to the lumped mass approximation, however, better correlation could probably have been achieved if more elements had been included in the dynamic model.

Similar results are shown in Fig. 3 for the second example which depicts the reverse situation of lowering the fundamental frequency from 242 to 200 cps. The initial element configuration for this problem was the same as for the previous example except each tube had a thickness of 0.02 in.

The final two examples represent the application of the computer program to complex structures. The two configurations treated are typical of today's spacecraft. Presented in Fig. 4 are the redesign steps required to raise the fundamental frequency of a frame structure, having 126 degrees of freedom, 30% above its initial value. Optimization was performed on 56 out of a total of 67 structural elements. Four cycles in the frequency-modification mode were necessary to reach the low end of the desired frequency band which represents frequencies within one percent of the desired value. The remaining redesign cycles occur in the weight-minimization mode except for the fifth and the eighth which represent steps to return the frequency to the acceptable band. In all examples treated to date there has been a consistent tendency for the frequency to drift downward in the weight-minimization mode. Consequently, an occasional shift to the frequency modification-mode is necessary to maintain the desired frequency level. The solution was manually stopped after ten cycles since it appeared that the 30% weight increase of the variable elements to achieve the same percentage increase in frequency was close to the optimum. For this particular example, it was felt that the solution was adequate even though the convergence criteria had not been met.

The substructure analysis capability was utilized in the next example which had 246 degrees of freedom. The dynamic model was comprised of three components containing a total of 55 structural elements. The objective of the analysis was to minimize the weight of nine critical elements while maintaining the existing frequency level. As indicated in Fig. 5, a potential weight reduction of 7.4 lb was achieved in 8 optimization cycles. This represents more than a 22% weight savings since the initial weight of the nine variable elements was 33.5 lb.

5. Conclusions

The method of analysis presented represents the first published procedure for redesigning a complex structure to achieve the least weight configuration which must have its fundamental frequency equal to some specified value. A computer program is described which completely automates the redesign process. A substructure analysis capability is included to provide the analyst with a practical optimization program for treating today's large structures. The feasibility of the approach is demonstrated in numerical examples which show the procedure to be convergent and capable of determining optimized configurations in as few as 10 redesign cycles.

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